Amalg. Worksheet # 3 Solutions

Various Artists

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1 Mike Hartglass

1.) Do the following formulae define inner products on the given vector spaces? (here $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{C}^2$

a.) $V = \mathbb{C}^2$, $\langle x, y\rangle = x_1 y_1 + x_2 y_2$

Solution: This is not an inner product. It is easy to see directly that $\langle x, y\rangle$ need not be equal to $\overline{\langle y, x\rangle}$ (for example if $x = (1, i) = y$).

b.) $V = \mathbb{C}^2$, $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2}$

Solution: This is an inner product. The verifications are left to you

c.) $V = \mathbb{C}^2$, $\langle x, y\rangle = x_1\overline{y_2} + x_2\overline{y_1}$

Solution: This is not an inner product. For example, if $x = (1, 0)$ then it is easy to see that $\langle x, x\rangle = 0$ even though x is nonzero.

d.) $V = \mathcal{P}^2(\mathbb{C})$, $\langle p, q\rangle = p(0)\overline{q(0)} + p(\sqrt{2})\overline{q(\sqrt{2})} + p(\pi)\overline{q(\pi)}$

Solution: This is an inner product on V. The linearity properties are left for you to verify. To show positive definiteness, notice that if $\langle p, p, \rangle = 0$, then $|p(0)|^2 + |p(\sqrt{2})|^2 + |p(\pi)|^2 = 0$. As each term is nonnegative, it follows that each term in the sum is zero, i.e. p hs at least three roots. Since p is a polynomial of degree at most 2, it follows that $p = 0$.

(*Remark*: Do you see why this is *not* an inner product on $\mathcal{P}_3(\mathbb{C})$?

2.) Suppose u and v are nonzero vectors in an inner product space v.

$a.$) Define

$$y = \frac{\langle v, w\rangle}{\langle w, w\rangle} w \text{ and } z = v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w.$$

Show that $v = y + z$, $y \in \text{span}(w)$, and z is orthogonal to every vector in span(w).

The verification that $v = y + z$ is trivial, and as $\frac{\langle v,w\rangle}{\langle w,w\rangle}$ is a scalar, it follows that y is in the span of w (this expression is the orthogonal projection of v onto the span of w). Finally, we see that

$$\langle z, w\rangle = \left\langle \left(v - \frac{\langle v, w\rangle}{\langle w, w\rangle} w\right), w \right\rangle = \langle v, w\rangle - \frac{\langle v, w\rangle}{\langle w, w\rangle} \cdot \langle w, w\rangle = 0.$$

Therefore, $\langle z, aw\rangle = \bar{\langle} z, w\rangle = 0$ for all $a \in \mathbb{F}$.

$b.$) Draw a picture of this in $\mathbb{R}^2$ for $w = (1, 0)$ and $v = (1, 1)$.

Solution: This is up to you to do

3.) Suppose $(e_1, ..., e_n)$ is an orthonormal basis for a vector space V, and let $x = c_1e_1 + \cdots + c_ne_n$. Find a formula for the c_i's.

Solution: We see that

$$\langle x, e_i\rangle = \langle c_1e_1 + \cdots + c_ne_n, e_i\rangle = c_1\langle e_1, e_i\rangle + \cdots + c_n\langle e_n, e_i\rangle$$

Using orthonormality, ($\langle e_i, e_j\rangle = 0$ is $i \neq j$ and $\langle e_i, e_1\rangle = 1$), we obtain $c_i = \langle x, e_i\rangle$.
4.) $a.$) Suppose x and y are orthogonal vectors in an inner product space V. Prove that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

Solution: We see that

$$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x + y\rangle + \langle y, x + y\rangle$$

$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + 0 + 0 + \|y\|^2 = \|x\|^2 + \|y\|^2$$

$b.$) Suppose x and y are vectors in an inner product space V. Prove that

$$\|x + ay\| \geq \|x\| \text{ for all } a \in \mathbb{F} \text{ if and only if } \langle x, y\rangle = 0.$$

Draw a picture of this in $\mathbb{R}^2$.

Solution: If x is orthogonal to y then x is orthogonal to ay for all $a \in \mathbb{F}$, so from part $a.$),

$$\|x + ay\|^2 = \|x\|^2 + \|ay\|^2 \geq \|x\|^2$$

so $\|x + ay\| \geq \|x\|$. Conversely, suppose x is not orthogonal to y (so in particular $\|y\| \neq 0$). Notice that we have the formula

$$\|x + ay\|^2 = \|x\|^2 + a\langle y, x\rangle + \overline{a}\langle x, y\rangle + |a|^2\|y\|^2 = \|x\|^2 + 2\Re(a \cdot \langle x, y\rangle) + |a|^2\|y\|^2$$

Therefore, we choose $a \in \mathbb{F}$ such that $a\langle x, y\rangle$ is real and strictly negative (so $2\Re(a \cdot \langle x, y\rangle) = 2a\langle x, y\rangle$), and $0 < |a| < 2\frac{|\langle x,y\rangle|}{\|y\|^2}$. Notice that the condition on a implies that the terms $2a\langle x, y\rangle$ and $|a|^2\|y\|^2$ have opposite signs and $|2a\langle x, y\rangle| > |a|^2\|y\|^2$. This implies, from the above expression for $\|x + ay\|^2$ that

$$\|x + ay\|^2 < \|x\|^2.$$

2 Peyam Tabrizian

Problem 1:

Suppose $\langle , \rangle$ is an inner product on W, and $T : V \to W$ is injective. Show that:

$$(u, v) := \langle T(u), T(v)\rangle$$

is an inner product on V.

Solution:

(a)

$$\begin{aligned}(u + w, v) &= < T(u + w), T(v) > \\ &= < T(u) + T(w), T(v) > \\ &= < T(u), T(v) > + < T(w), T(v) > \\ &= (u, v) + (w, v)\end{aligned}$$

And:

$$\begin{aligned}(au, v) &= < T(au), T(v) > \\ &= < aT(u), T(v) > \\ &= a < T(u), T(v) > \\ &= a(u, v)\end{aligned}$$

(b)

$$\begin{aligned}(v,u) &= <T(v),T(u)> \\ &= \overline{<T(u),T(v)>} \\ &= \overline{(u,v)}\end{aligned}$$

(c)

$$(u,u) =< T(u), T(u) > \ \geq 0$$

Moreover, if $(u,u)=0$, then $<T(u),T(u)>=0$, so $T(u)=0$, so $u=0$ since T is injective. □

Problem 2:

Show that if $v_1, \cdots, v_k$ are nonzero orthogonal vectors, then $(v_1, \cdots, v_k)$ is linearly independent.

Solution: Suppose:

$$a_1 v_1 + \cdots + a_k v_k = 0 \quad (*)$$

Fix $i = 1, \cdots, k$ and take the inner product of $(*)$ with v_i:

$$\begin{aligned} < a_1 v_1 + \cdots + a_k v_k, v_i > = &\ 0 \\ a_1 < v_1, v_i > + \cdots + a_i < v_i, v_i > + \cdots + a_k < v_k, v_i > = &\ 0 \\ a_1 0 + \cdots + a_i < v_i, v_i > + \cdots + a_k 0 = &\ 0 \\ a_i < v_i, v_i > = &\ 0 \\ a_i = &\ 0 \end{aligned}$$

Where in the third equality, we used the fact that $v_1, \cdots, v_k$ are orthogonal, and in the last inequality, we used $v_i \neq 0$, so $< v_i, v_i >= \|v_i\|^2 > 0$

So $a_1 = \cdots = a_k = 0$, since i was arbitrary □

Problem 3:

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Show that every eigenvalue of T is real.

Solution: Suppose $T(v) = \lambda v$, for $v \neq 0$.

Consider $< T(v), v >$.

On the one hand:

$$< T(v), v >=< \lambda v, v >= \lambda < v, v >= \lambda \|v\|^2$$

On the other hand:

$$< T(v), v >=< v, T^*(v) >=< v, T(v) >=< v, \lambda v >= \overline{\lambda} < v, v >= \overline{\lambda} \|v\|^2$$

(where we used the definition of T^* and the fact that $T^* = T$ because T is self-adjoint)
Hence:

$$\lambda \|v\|^2 = \overline{\lambda} \|v\|^2$$

So $\boxed{\lambda = \overline{\lambda}}$, because $\|v\| > 0$, since $v \neq 0$ $\square$

Problem 4:

Show that if T is normal, then $Nul(T^*) = Nul(T)$

Solution: Suppose $v \in Nul(T)$, then $T(v) = 0$, so $T^*T(v) = T^*(T(v)) = T^*(0) = 0$.

Hence:

$$\begin{aligned}
0 &= < 0, v > \\
&= < T^*Tv, v > \\
&= < TT^*v, v > && \text{because } T \text{ is normal, so } T^*T = TT^* \\
&= < T^*v, T^*v > \\
&= \|T^*v\|^2
\end{aligned}$$

Hence $\|T^*v\|^2 = 0$, hence $T^*v = 0$, so $v \in Nul(T^*)$

Hence $\boxed{Nul(T) \subseteq Nul(T^*)}$.

In particular, notice that $(T^*)^*T^* = TT^* = T^*T = T^*(T^*)^*$, so T^* is normal, and hence by what we've just shown:
$\boxed{Nul(T^*) \subseteq Nul((T^*)^*) = Nul(T)}$.

Hence $Nul(T^*) = Nul(T)$ $\square$

Problem 5:

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V.
Show that U is invariant under T if and only if $U^\perp$ is invariant under T^*

Solution:
($\Rightarrow$) Suppose $v \in U^\perp$, want to show $T^*(v) \in U^\perp$.

But for all $u \in U$:
$< T^*v, u >=< v, T(u) >= 0$, since $T(u) \in U$ (since U is T-invariant)and $v \in U^\perp$
So $T^*v \in U^\perp$ by definition of $U^\perp$

($\Leftarrow$) $U^\perp$ invariant under T^* implies $(U^\perp)^\perp$ is invariant under $(T^*)^*$.

However, $(U^\perp)^\perp = U$ (here we use the fact that V is **finite-dimensional**) and $(T^*)^* = T$

Hence we get: U is invariant under T. $\square$

Problem 6:

(if time permits) Suppose V is finite-dimensional and U is a subspace of V.
Show that $V = U \oplus U^\perp$

Solution: We'll show[1]:

[1] This is enough, because if $(u_1, \cdots, u_k)$ is a basis of U and $(w_1, \cdots, w_l)$ is a basis of $U^\perp$, you can show using (a) and (b) that $(u_1, \cdots, w_l)$ is a basis of V, and hence $V = Span(u_1, \cdots, w_l) = Span(u_1, \cdots, u_k) + Span(w_1, \cdots, w_l) = U + U^\perp$. And then use (a) and Prop 1.9

(a) $U \cap U^{\perp} = \{0\}$

(b) $dim(V) = dim(U) + dim(U^{\perp})$

(a) Suppose $u \in U \cap U^{\perp}$

Then $< u, u >= 0$, because $u \in U$ and $u \in U^{\perp}$, by definition of $U^{\perp}$.

Hence $\|u\|^2 = 0$, so $u = 0$

(b) Let $(u_1, \cdots, u_k)$ be an **orthonormal** basis of U[2]. Extend this to an orthonormal basis $(u_1, \cdots, u_k, w_1, \cdots, w_l)$ of V[3].

Claim: $(w_1, \cdots, w_l)$ is a basis of $U^{\perp}$

Then we're done, because $dim(V) = k + l = dim(U) + dim(U^{\perp})$.

Proof: Linear independence follows from Problem 2 because $w_1, \cdots, w_l$ are nonzero orthogonal vectors.

Let $W = Span(w_1, \cdots, w_l)$, we'll show $W \subseteq U^{\perp}$ and $U^{\perp} \subseteq W$.

$\boxed{W \subseteq U^{\perp}}$ If $u \in U$, then $u = a_1 u_1 + \cdots + a_k u_k$ for scalars $a_1, \cdots, a_k$ (because $(u_1, \cdots, u_k)$ is a basis of U).

But then for every $i = 1, \cdots, l$:

$$\begin{aligned} < w_i, u > &= < w_i, a_1 u_1 + \cdots + a_k u_k > \\ &= a_1 < w_i, u_1 > + \cdots + a_k < w_i, u_k > \\ &= a_1 0 + \cdots + a_k 0 \\ &= 0 \end{aligned}$$

Where we used the fact that $(u_1, \cdots, w_l)$ is orthogonal.

[2] Orthogonal also works

[3] Such a basis exists by Corollary 6.25. Orthogonal also works

Hence each $w_i \in U^\perp$, and hence $W = Span(w_1, \cdots, w_l) \subseteq U^\perp$.

$\boxed{U^\perp \subseteq W}$ If $v \in U^\perp$, then $< v, u >= 0$ for all $u \in U$, and in particular, $< v, u_i >= 0$ for all $i = 1, \cdots, k$.

Since $(u_1, \cdots, w_l)$ is a basis for V and $v \in V$, $v = a_1 u_1 + \cdots + a_k u_k + b_1 w_1 + \cdots + b_l w_l$ for scalars $a_1, \cdots, b_l$.

But then for all $i = 1, \cdots, k$:

$$\begin{aligned} 0 &= < v, u_i > \\ &= < a_1 u_1 + \cdots + a_k u_k + b_1 w_1 + \cdots + b_l w_l, u_i > \\ &= a_1 < u_1, u_i > + \cdots + a_i < u_i, u_i > + \cdots + a_k < u_k, u_i > + b_1 < w_1, u_i > + \cdots + b_l < w_l, u_i > \\ &= a_1 0 + \cdots + a_i 1 + \cdots + a_k 0 + b_1 0 + \cdots + b_l 0 \\ &= a_i \end{aligned}$$

Hence $a_i = 0$ for all $i = 1, \cdots, k$, and hence:

$$v = a_1 u_1 + \cdots + a_k u_k + b_1 w_1 + \cdots + b_l w_l = b_1 w_1 + \cdots + b_l w_k \in Span(w_1, \cdots, w_l) = W$$

Hence $U^\perp \subseteq W$ □

Problem 7:

(if time permits) Let $(v_1, \cdots, v_n)$ be an orthonormal basis of V and suppose the matrix of $T \in \mathcal{L}(V)$ is A. What is the matrix of T^* with respect to that same basis?

Solution: Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}$, where $i = 1, \cdots, n$, $j = 1, \cdots, n$.

To find the matrix of T^*, as usual, for all $j = 1, \cdots, n$, calculate $T^*(v_j)$ and then express the result in terms of $v_1, \cdots, v_n$.

Before we do that, **notice** that if $w = b_1 v_1 + \cdots + b_n v_n$, then for all $i = 1, \cdots, n$,

$$\begin{aligned} < w, v_i > &= < b_1 v_1 + \cdots + b_n v_n, v_i > \\ &= b_1 < v_1, v_i > + \cdots + b_i < v_i, v_i > + \cdots + b_n < v_n, v_i > \\ &= b_1 0 + \cdots + b_i 1 + \cdots + b_n 0 \\ &= b_i \end{aligned}$$

Where we used the fact that $v_1, \cdots, v_n$ are orthonormal.

The point is that $< w, v_i >$ directly gives you the i-th coefficient in the expression of w as a linear combo of $v_1, \cdots, v_n$.[4]

In particular, taking $w = T^*(v_j)$, we get that $< T^*(v_j), v_i >$ gives you the $i-$th coefficient in the expression of $T^*(v_j)$ as a linear combo of $v_1, \cdots, v_n$. In other words, $< T^*(v_j), v_i >$ gives you the $(i, j)-$th entry of the matrix of T^* with respect to the basis $(v_1, \cdots, v_n)$!

However:

$$\begin{aligned} < T^*(v_j), v_i > &= < v_j, T(v_i) > \\ &= < v_j, a_{1i} v_1 + \cdots + a_{ji} v_j + \cdots + a_{ni} v_n > \quad \text{by definition of } A\text{, the matrix of } T \\ &= < v_j, a_{1i} v_1 > + \cdots + < v_j, a_{ji} v_j > + \cdots + < v_j, a_{ni} v_n > \\ &= \overline{a_{1i}} < v_j, v_1 > + \cdots + \overline{a_{ji}} < v_j, v_j > + \cdots + \overline{a_{ni}} < v_j, v_n > \\ &= \overline{a_{1i}}\, 0 + \cdots + \overline{a_{ji}}\, 1 + \cdots + \overline{a_{ni}}\, 0 \quad \text{by orthonormality} \\ &= \overline{a_{ji}} \end{aligned}$$

Hence, by the above, we have $(A^*)_{ij} = \overline{a_{ji}} = \overline{(A)_{ji}}$, that is:

$$A^* = \begin{bmatrix} \overline{a_{11}} & \cdots & \overline{a_{n1}} \\ \vdots & & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{nn}} \end{bmatrix}$$

That is, A^* is (in fact), the **conjugate transpose** of A. □

Note: In particular, if $\mathbb{F} = \mathbb{R}$, then $A^* = A^T$

[4]this is what makes orthonormality so awesome!

3 Daniel Sparks

3.1

Let $u = \sum_{i=1}^{m} a_i u_i \in U$ and $w = \sum_{j=1}^{k} b_j w_j \in W$ be arbitrary. Then

$$\begin{aligned}
\left\langle \sum_{i=1}^{m} a_i u_i, \sum_{j=1}^{k} b_j w_j \right\rangle &= \sum_{i=1}^{m} a_i \left\langle u_i, \sum_{j=1}^{k} b_j w_j \right\rangle \\
&= \sum_{i=1}^{m} a_i \left(\sum_{j=1}^{k} \overline{b_j} \langle u_i, w_j \rangle \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{k} (a_i \overline{b_j})(0) \\
&= 0
\end{aligned}$$

There are other ways to do this, using common useful facts, but you'd have to prove them if they weren't presented in your book. For example $U \perp (W_1 + \cdots + W_k)$ if and only if $U \perp W_i$ for $i = 1, \cdots, n$, whose proof is immediate. Also, $\mathbf{F}v \perp \mathbf{F}w$ (where $v, w \neq 0$, if and only if $v \perp w$. Combining these, and using induction, gives another proof.

3.2

Let $\dim V = n$. We have seen in a previous homework exercise that $\text{Null}(P^n) \oplus \text{Range}(P^n) = V$, we actually have that $\text{Null}(P) \oplus \text{Range}(P) = V$. This is because $P^k = P$ for all $k \geq 1$. [Induction: the base case $k = 1$ is clear by definition. Suppose $P^k = P$, then $P^{k+1} = P \circ P^k = P \circ P = P^2 = P$, completing the induction.] Let $\{v_1, \cdots, v_r\}$ be a basis for $\text{Range}(P)$ and $\{v_{r+1}, \cdots, v_n\}$ be a basis for $\text{Null}(P)$. Take $\beta = \{v_1, \cdots, v_n\}$ as basis for V.

Recall that P is self adjoint if and only if $\langle Pv, w \rangle = \langle v, Pw \rangle$ for all $v, w \in V$. We observe that it is sufficient to check this on a basis:

Lemma: For any basis $\beta = \{v_1, \cdots, v_n\}$, P is self adjoint if and only if $\langle Pv_i, v_j \rangle = \langle v_i, Pv_j \rangle$ for $1 \leq i, j \leq n$.

Solution to exercise: Suppose that P is self adjoint, and consider v_i with $i \leq r$ (i.e., $v_i \in \text{Range}(P)$) and v_j with $j > r$ (i.e. $v_j \in \text{Null}(P)$). Then

$$\langle v_i, v_j = \langle Pv_i, v_j \rangle = \langle v_i, Pv_j \rangle = \langle v_i, 0 \rangle = 0$$

That means that for any v_i, v_j with $i \le r < j$, we have $v_i \perp v_j$. Hence by Exercise 1, $\text{Range}(P) \perp \text{Null}(P)$.

Conversely, suppose $\text{Range}(P) \perp \text{Null}(P)$. Let i, j be any two numbers such that $1 \le i \le n$ and $1 \le j \le n$. We consider four cases:

1. $i \le r, j \le r$. Then both $v_i, v_j \in \text{Range}(P)$. Hence $\langle Pv_i, v_j\rangle = \langle v_i, v_j\rangle = \langle v_i, Pv_j\rangle$.

2. $i \le r, j > r$. Then $v_i \in \text{Range}(P)$ but $v_j \in \text{Null}(P)$. Then $\langle Pv_i, v_j\rangle = \langle v_i, v_j\rangle = 0$ by assumption. On the other hand $0 = \langle v_i, 0\rangle = \langle v_i, Pv_j\rangle$. So $\langle Pv_i, v_j\rangle = 0 = \langle v_i, Pv_j\rangle$.

3. $i > r, j \le r$. Then $v_i \in \text{Null}(P)$ but $v_j \in \text{Range}(P)$. Then, again, $\langle Pv_i, v_j\rangle = \langle 0, v_j\rangle = 0 = \langle v_i, v_j\rangle = \langle v_i, Pv_j\rangle$.

4. $i > r, j > r$. Then $v_i, v_j \in \text{Null}(P)$. Then $\langle Pv_i, v_j\rangle = \langle 0, v_j\rangle = 0 = \langle v_i, 0\rangle = \langle v_i, Pv_j\rangle$.

In each case we see that $\langle Pv_i, v_j\rangle = \langle v_i, Pv_j\rangle$. By the Lemma, P is self-adjoint. This wraps it up unless the lemma has not yet been covered in class or the book. $\square$

Proof of Lemma: (My way:) The sesquilinear map $V \times V \to \mathbf{C}$ by $(v, w) \mapsto \langle Pv, w\rangle - \langle v, Pw\rangle$ is determined by its values on the basis (v_i, v_j) for $1 \le i, j \le n$.

(Peyam might prefer:) The "only if" is clear, as the definition of self adjoint is quantified over arbitrary v, w, simply take $v = v_i$, $w = v_j$. For the other direction, suppose $\langle Pv_i, v_j\rangle - \langle v_i, Pv_j\rangle = 0$ for $1 \le i, j \le n$. Now let $v = \sum_{i=1}^{n} a_i v_i$ and $w = \sum_{j=1}^{n} b_j v_j$ be arbitrary vectors in V. Then

$$\begin{aligned}
\langle Pv, w\rangle - \langle v, Pw\rangle &= \left\langle P\left(\sum_{i=1}^{n} a_i v_i\right), \sum_{j=1}^{n} b_j v_j\right\rangle - \left\langle \sum_{i=1}^{n} a_i v_i, P\left(\sum_{j=1}^{n} b_j v_j\right)\right\rangle \\
&= \left\langle \sum_{i=1}^{n} a_i P v_i, \sum_{j=1}^{n} b_j v_j\right\rangle - \left\langle \sum_{i=1}^{n} a_i v_i, \sum_{j=1}^{n} b_j P v_j\right\rangle \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{b_i}\langle Pv_i, v_j\rangle\right) - \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \overline{b_j}\langle v_i, Pv_j\rangle\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \overline{b_j}(\langle Pv_i, v_j\rangle - \langle v_i, Pv_j\rangle) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} (a_i \overline{b_j})(0) \\
&= 0
\end{aligned}$$